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General Jacobi Identity Revisited

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In a previous paper (Nishimura, 1997) we probed the deeper structure of the Jacobi identity of vector fields with respect to Lie brackets within the realm of synthetic differential geometry to find what might be called the general Jacobi identity of microcubes. The main objective of this paper is to present a less esoteric and more lucid proof of it.

INTRODUCTION

Kock and Lavendhomme (1984) have developed a theory of microsquares in which Lie brackets of vector fields on a microlinear space M can be expressed as strong differences of their associated microsquares on M^M. Nishimura (1997) took a step forward to find that the Jacobi identity of vector fields on M with respect to Lie brackets is reverberation of a deeper and more fundamental identity of microcubes on M^M which might be called the *general Jacobi identity.* Though its proof there was thoroughly correct and exact, the exposition might appear somewhat precipitous and esoteric. The principal objective of this paper is to elaborate it into a less esoteric and more comprehensible one.

The main text of the paper consists of three sections, the first two of which are a brief review of Kock and Lavendhomme (1984) and Nishimura (1997) and are intended mainly to fix our notation and prepare the reader for more advanced quasi-colimit diagrams in the last section. The first and second sections are devoted to simplicial objects and strong differences, respectively. The gigantic quasi-colimit diagram of small objects in our previous paper (Nishimura, 1997, Lemma 3.3) is successfully divided into a few more manageable and more accessible ones in the last section. In particular,

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the core of the proof of the general Jacobi identity will be crystallized into an elegant quasi-colimit diagram of small objects in Theorem 3.5.

We assume that the reader is familiar with Lavendhomme's (1996) readable textbook on synthetic differential geometry up to Chapter 3. We choose once and for all, a microlinear space M, which shall be fixed throughout the paper. The extended set of real numbers including infinitesimal ones is denoted by $\mathbb{R}$ and is expected to satisfy the general Kock axiom. We denote $\{d \in \mathbb{R} | d^2 = 0\}$ by D as usual. Elements of D are usually denoted by d with or without subscripts. As is usual in synthetic differential geometry, the reader should presume that we are working in a non-Boolean topos, so that the principle of excluded middle and Zorn's lemma should be avoided. But for these two points, we would feel that we are working in the standard universe of sets.

1. SIMPLICIAL OBJECTS

In this section we distinguish a clear-cut class of small objects. Let n be a natural number and $\mathtt{n}$ the set consisting exactly of $1, 2, \ldots, n$. Let Δ_n be the set of finite sequences $(i_1, \ldots, i_k)$ in $\mathtt{n}$ with $i_1 < \cdots < i_k$ and $k \geq 2$. Given a finite subset $\mathtt{p}$ of Δ_N, we define a small object $D^n\{\mathtt{p}\}$ as follows:

$$(1.1) \quad D^n\{\mathtt{p}\} = \{(d_1, \ldots, d_n) \in D^n | d_{i_1} \ldots d_{i_k} = 0 \text{ for any } (i_1, \ldots, i_k) \in \mathtt{p}\}$$

If $\mathtt{p}$ is empty, $D^n\{\mathtt{p}\}$ is D^n itself. If $\mathtt{p}$ is Δ_n, then $D^n\{\mathtt{p}\}$ is $D(n)$ in standard terminology. Small objects of the form $D^n\{\mathtt{p}\}$ are called *simplicial objects of degree n*. If $\mathtt{p} \subset \mathtt{q} \subset \Delta_n$, then $D^n\{\mathtt{q}\}$ is a subset of $D^n\{\mathtt{p}\}$, in which the canonical injection of $D^n\{\mathtt{q}\}$ into $D^n\{\mathtt{p}\}$ is generally denoted by $\mathtt{i}$. Given two simplicial objects $D^m\{\mathtt{p}\}$ and $D^n\{\mathtt{q}\}$ of degrees m and n, respectively, we define a simplicial objects $D^m\{\mathtt{p}\} \oplus D^n\{\mathtt{q}\}$ to be $D^{m+n}\{\mathtt{p} \oplus \mathtt{q}\}$, where

$$(1.2) \quad \mathtt{p} \oplus \mathtt{q} = \mathtt{p} \cup \{(j_1 + m, \ldots, j_k + m) | (j_1, \ldots, j_k) \in \mathtt{q}\} \cup \{(i, j+m) | 1 \leq i \leq m, 1 \leq j \leq n\}$$

By way of example, $D(m) \oplus D(n)$ is $D(m+n)$. Simplicial objects $D^m\{\mathtt{p}\}$ and $D^n\{\mathtt{q}\}$ can naturally be regarded as subsets of $D^m\{\mathtt{p}\} \oplus D^n\{\mathtt{q}\}$. Since the operation $\oplus$ is associative, we can combine any finite number of simplicial objects by $\oplus$ without bothering about how to insert parentheses. Given morphisms of simplicial objects (i.e., mappings induced by their corresponding homomorphisms of Weil algebras) $\beta_i: D^{m_i}\{\mathtt{p}_i\} \to D^m\{\mathtt{p}\}$ $(1 \leq i \leq n)$, there exists a unique function $\beta: D^{m_1}\{\mathtt{p}_1\} \oplus \cdots \oplus D^{m_n}\{\mathtt{p}_n\} \to D^m\{\mathtt{p}\}$ whose restriction to $D^{m_i}\{\mathtt{p}_i\}$ coincides with β_i for each i. We denote this β by $\beta_1 \oplus \cdots \oplus \beta_n$.

Given a simplicial object $D^n\{\mathtt{p}\}$, we denote by $\mathsf{T}^{D^n\{\mathtt{p}\}}(M)$ the set of all functions from $D^n\{\mathtt{p}\}$ to M. In particular, $\mathsf{T}^D(M)$ is the set of tangent vectors to M, $\mathsf{T}^{D^2}(M)$ is the set of microsquares on M, and $\mathsf{T}^{D^3}(M)$ is the set of microcubes on M. It is not difficult to see that, given mappings γ_i: $D^{m_i}\{\mathtt{p}_i\} \to M$ $(1 \le i \le n)$ with $\gamma_1(0) = \cdots = \gamma_n(0)$, there exists a unique mapping $\ell_{(\gamma_1,\ldots,\gamma_n)}$: $D^{m_1}\{\mathtt{p}_1\} \oplus \cdots \oplus D^{m_n}\{\mathtt{p}_n\} \to M$ whose restriction to each ith axis coincides with γ_i.

We note in passing that Lavendhomme and Nishimura (1998) have developed a synthetic theory of differential forms based on simplicial objects. The general theory of simplicial objects in synthetic differential geometry will be discussed elsewhere.

2. STRONG DIFFERENCES

The following proposition is taken from Lavendhomme (1996, §3.4).

Proposition 2.1. The diagram

$$\begin{array}{ccc} D(2) & \xrightarrow{\ \mathtt{i}\ } & D^2 \\ {\scriptstyle \mathtt{i}}\downarrow & & \downarrow{\scriptstyle \psi} \\ D^2 & \xrightarrow[\varphi]{} & D^3\{(1,3),(2,3)\} \end{array}$$

is a quasi-colimit diagram of small objects, where

(2.1) $\varphi(d_1, d_2) = (d_1, d_2, 0)$
(2.2) $\psi(d_1, d_2) = (d_1, d_2, d_1 d_2)$

for any $(d_1, d_2) \in D^2$.

As a direct corollary of the above proposition we have:

Proposition 2.2. For any $\gamma_1, \gamma_2 \in \mathsf{T}^2(M)$, if $\gamma_1|_{D(2)} = \gamma_2|_{D(2)}$, then there exists a unique γ: $D^3\{(1, 3), (2, 3)\} \to M$ with $\gamma \circ \varphi = \gamma_1$ and $\gamma \circ \psi = \gamma_2$.

We will write $\mathtt{g}_{(\gamma_1,\gamma_2)}$ for γ in the above proposition. The strong difference $\gamma_2 \dot{-} \gamma_1$ is defined to be the tangent vector $d \in D \mapsto \mathtt{g}_{(\gamma_1,\gamma_2)}(0, 0, d)$ to M.

By relativizing Proposition 2.1 we have:

Proposition 2.3. The diagram

$$\begin{array}{ccc} D^3\{(2,3)\} & \xrightarrow{\ \mathtt{i}\ } & D^3 \\ {\scriptstyle \mathtt{i}}\downarrow & & \downarrow{\scriptstyle \psi_1^3} \\ D^3 & \xrightarrow[\varphi_1^3]{} & D^4\{(2,4),(3,4)\} \end{array}$$

is a quasi-colimit diagram of small objects, where

(2.3) $\varphi_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$
(2.4) $\psi_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_2d_3)$

for any $(d_1, d_2, d_3) \in D^3$.

As a direct corollary of the above proposition we have:

Proposition 2.4. For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M)$, if $\gamma_1|_{D^3\{(2,3)\}} = \gamma_2|_{D^3\{(2,3)\}}$, then there exists a unique $\gamma: D^4\{(2, 4), (3, 4)\} \to M$ with $\gamma \circ \varphi_1^3 = \gamma_1$ and $\gamma \circ \psi_1^3 = \gamma_2$.

We will write $\mathfrak{g}^1_{(\gamma_1,\gamma_2)}$ for γ in the above proposition. The strong difference $\gamma_2 \dot{\underset{1}{-}} \gamma_1$ is defined to be the microsquare $(d_1, d_2) \in D^2 \mapsto \mathfrak{g}^1_{(\gamma_1,\gamma_2)}(d_1, 0, 0, d_2)$ on M.

An appropriate variant of Proposition 2.3 readily yields:

Proposition 2.5. For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M)$, if $\gamma_1|_{D^3\{(1,3)\}} = \gamma_2|_{D^3\{(1,3)\}}$, then there exists a unique $\gamma: D^4\{(1, 4), (3, 4)\} \to M$ with $\gamma \circ \varphi_2^3 = \gamma_1$ and $\gamma \circ \psi_2^3 = \gamma_2$, where functions $\varphi_2^3, \psi_2^3: D^3 \to D^4\{(1, 4), (3, 4)\}$ go as follows:

(2.5) $\varphi_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$
(2.6) $\psi_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1d_3)$

We will write $\mathfrak{g}^2_{(\gamma_1,\gamma_2)}$ for γ in the above proposition. The strong difference $\gamma_2 \dot{\underset{2}{-}} \gamma_1$ is defined to be the microsquare $(d_1, d_2) \in D^2 \mapsto \mathfrak{g}^2_{(\gamma_1,\gamma_2)}(0, d_1, 0, d_2)$ on M.

Again an appropriate variant of Proposition 2.3 readily yields:

Proposition 2.6. For any $\gamma_1, \gamma_2 \in \mathsf{T}^3(M)$, if $\gamma_1|_{D^3\{(1,2)\}} = \gamma_2|_{D^3\{(1,2)\}}$, then there exists a unique $\gamma: D^4\{(1, 4), (2, 4)\} \to M$ with $\gamma \circ \varphi_3^3 = \gamma_1$ and $\gamma \circ \psi_3^3 = \gamma_2$, where functions $\varphi_3^3, \psi_3^3: D^3 \to D^4\{(1, 4), (2, 4)\}$ go as follows:

(2.7) $\varphi_3^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0)$
(2.8) $\psi_3^3(d_1, d_2, d_3) = (d_1, d_2, d_3, d_1d_2)$

We will write $\mathfrak{g}^3_{(\gamma_1,\gamma_2)}$ for γ in the above proposition. The strong difference $\gamma_2 \dot{\underset{3}{-}} \gamma_1$ is defined to be the microsquare $(d_1, d_2) \in D^2 \mapsto \mathfrak{g}^2_{(\gamma_1,\gamma_2)}(0, 0, d_1, d_2)$ on M.

The general Jacobi identity goes as follows:

Theorem 2.7. Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in \mathsf{T}^3(M)$. As long as the following three expressions are well defined, they sum up only to vanish:

(2.9) $(\gamma_{123} \dot{\underset{1}{-}} \gamma_{132}) \dot{-} (\gamma_{231} \dot{\underset{1}{-}} \gamma_{321})$
(2.10) $(\gamma_{231} \dot{\underset{2}{-}} \gamma_{213}) \dot{-} (\gamma_{312} \dot{\underset{2}{-}} \gamma_{132})$
(2.11) $(\gamma_{312} \dot{\underset{3}{-}} \gamma_{321}) \dot{-} (\gamma_{123} \dot{\underset{3}{-}} \gamma_{213})$

For the relationship of the above identity to the well-known Jacobi identity of vector fields, the reader is referred to Nishimura (1997, Theorem 3.2).

3. THE GENERAL JACOBI IDENTITY

This section is devoted completely to a proof of Theorem 2.7. Let us begin with the following result:

Proposition 3.1. The diagram

$$\begin{array}{ccc} D(2) & \xrightarrow{\mathrm{i}_{14}^{1}} & D^4\{(2,4),(3,4)\} \\ {\scriptstyle \mathrm{i}_{14}^{1}}\downarrow & & \downarrow{\scriptstyle \eta_2^1} \\ D^4\{(2,4),(3,4)\} & \xrightarrow[\eta_1^1]{} & E[1] \end{array}$$

is a quasi-colimit diagram of small objects with its quasi-colimit E, where:

(3.1) $E[1]$ is $D^7\{(2, 6), (3, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

(3.2) $\mathrm{i}_{14}^{1}(d_1, d_2) = (d_1, 0, 0, d_2)$ for any $(d_1, d_2) \in D(2)$.

(3.3) $\eta_1^1(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, 0, 0, d_4, 0)$ for any $(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\}$.

(3.4) $\eta_2^1(d_1, d_2, d_3, d_4) = (d_1, 0, 0, d_2, d_3, d_4, d_1d_4)$ for any $(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\}$.

Proof. The so-called general Kock axiom warrants that functions γ_1, γ_2: $D^4\{(2, 4), (3, 4)\} \to \mathbb{R}$ and γ: $E[1] \to \mathbb{R}$ should be polynomials of infinitesimals in D with coefficients in $\mathbb{R}$ of the following forms:

(3.5) $\gamma_1(d_1, d_2, d_3, d_4) = a + a_1d_1 + a_2d_2 + a_3d_3 + a_4d_4 + a_{12}d_1d_2 + a_{13}d_1d_3 + a_{14}d_1d_4 + a_{23}d_2d_3 + a_{123}d_1d_2d_3$

(3.6) $\gamma_2(d_1, d_2, d_3, d_4) = b + b_1d_1 + b_2d_2 + b_3d_3 + b_4d_4 + b_{12}d_1d_2 + b_{13}d_1d_3 + b_{14}d_1d_4 + b_{23}d_2d_3 + b_{123}d_1d_2d_3$

(3.7) $\gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = c + c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4 + c_5d_5 + c_6d_6 + c_7d_7 + c_{12}d_1d_2 + c_{13}d_1d_3 + c_{14}d_1d_4 + c_{15}d_1d_5 + c_{16}d_1d_6 + c_{23}d_2d_3 + c_{45}d_4d_5 + c_{123}d_1d_2d_3 + c_{145}d_1d_4d_5$

The condition that $\gamma_1 \circ \mathrm{i}_{14}^{1} = \gamma_2 \circ \mathrm{i}_{14}^{1}$ is equivalent to the following condition:

(3.8) $a = b$, $a_1 = b_1$, and $a_4 = b_4$.

Therefore it is not difficult to see that $\gamma_1 \circ \mathrm{i}_{14}^{1} = \gamma_2 \circ \mathrm{i}_{14}^{1}$ exactly when there exists γ: $E[1] \to \mathbb{R}$ with $\gamma \circ \eta_1^1 = \gamma_1$ and $\gamma \circ \eta_2^1 = \gamma_2$, in which γ is to be of the following form:

$$(3.9)\quad \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a + a_1d_1 + a_2d_2 + a_3d_3 + b_2d_4 + b_3d_5 + a_4d_6 + (b_{14} - a_{14})d_7 + a_{12}d_1d_2 + a_{13}d_1d_3 + b_{12}d_1d_4 + b_{13}d_1d_5 + a_{14}d_1d_6 + a_{23}d_2d_3 + b_{23}d_4d_5 + a_{123}d_1d_2d_3 + b_{123}d_1d_4d_5$$

This means that the above diagram is a quasi-colimit diagram of small objects. ■

We will write ι_1^1, ι_2^1, ι_3^1, and ι_4^1 for $\eta_1^1 \circ \varphi_1^3$, $\eta_1^1 \circ \psi_1^3$, $\eta_2^1 \circ \varphi_1^3$, and $\eta_2^1 \circ \psi_1^3$, respectively. That is to say, for any $(d_1, d_2, d_3) \in D^3$, we have

$$(3.10)\quad \iota_1^1(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0)$$
$$(3.11)\quad \iota_2^1(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_2d_3, 0)$$
$$(3.12)\quad \iota_3^1(d_1, d_2, d_3) = (d_1, 0, 0, d_2, d_3, 0, 0)$$
$$(3.13)\quad \iota_4^1(d_1, d_2, d_3) = (d_1, 0, 0, d_2, d_3, d_2d_3, d_1d_2d_3)$$

As a direct corollary of Propositions 2.3 and 3.1 we have:

Proposition 3.2. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathsf{T}^3(M)$, if the expression $(\gamma_4 \dot{\underset{1}{-}} \gamma_3) \dot{-} (\gamma_2 \dot{\underset{1}{-}} \gamma_1)$ is well defined, then there exists a unique $\gamma \in \mathsf{T}^{E[1]}(M)$ such that $\gamma \circ \iota_i^1 = \gamma_i$ $(i = 1, 2, 3, 4)$.

We will write $\mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ for γ is the above proposition. We note that for any $(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\}$,

$$(3.14)\quad \mathfrak{g}^1_{(\gamma_1,\gamma_2)}(d_1, d_2, d_3, d_4) = \mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(d_1, d_2, d_3, 0, 0, d_4, 0)$$
$$(3.15)\quad \mathfrak{g}^1_{(\gamma_3,\gamma_4)}(d_1, d_2, d_3, d_4) = \mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(d_1, 0, 0, d_2, d_3, d_4, d_1d_4)$$

Therefore, for any $(d_1, d_2) \in D^2$, we have

$$(3.16)\quad (\gamma_2 \dot{\underset{1}{-}} \gamma_1)(d_1, d_2) = \mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(d_1, 0, 0, 0, 0, d_2, 0)$$
$$(3.17)\quad (\gamma_4 \dot{\underset{1}{-}} \gamma_3)(d_1, d_2) = \mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(d_1, 0, 0, 0, 0, d_2, d_1d_2)$$

(3.16) and (3.17) imply that for any $(d_1, d_2, d_3) \in D^3\{(1, 3), (2, 3)\}$,

$$(3.18)\quad \mathfrak{g}_{(\gamma_2 \dot{\underset{1}{-}} \gamma_1, \gamma_4 \dot{\underset{1}{-}} \gamma_3)}(d_1, d_2, d_3) = \mathfrak{h}^1_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(d_1, 0, 0, 0, 0, d_2, d_3)$$

Therefore, for any $d \in D$, we have

$$(3.19)\quad ((\gamma_4 \dot{\underset{1}{-}} \gamma_3) \dot{-} (\gamma_2 \dot{\underset{1}{-}} \gamma_1))(d) = \mathfrak{h}^1_{(\gamma_2,\gamma_2,\gamma_3,\gamma_4)}(0, 0, 0, 0, 0, 0, d)$$

We will write $E[2]$ for $D^7\{(1, 6), (3, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (1, 4), (1, 5), (3, 4), (3, 5)\}$. We define functions ι_1^2, ι_2^2, ι_3^2, and ι_4^2 from D^3 to $E[2]$ as follows:

$$(3.20)\quad \iota_1^2(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0)$$
$$(3.21)\quad \iota_2^2(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_1d_3, 0)$$
$$(3.22)\quad \iota_3^2(d_1, d_2, d_3) = (0, d_2, 0, d_3, d_1, 0, 0)$$
$$(3.23)\quad \iota_4^2(d_1, d_2, d_3) = (0, d_2, 0, d_3, d_1, d_1d_3, d_1d_2d_3)$$

By the same token as in Proposition 3.2 we have:

Proposition 3.3. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathsf{T}^3(M)$, if the expression $(\gamma_4 \underset{2}{\div} \gamma_3) \dot{-} (\gamma_2 \underset{2}{\div} \gamma_1)$ is well defined, then there exists unique $\gamma \in \mathsf{T}^{E[2]}(M)$ such that $\gamma \circ \iota_i^2 = \gamma_i$ $(i = 1, 2, 3, 4)$.

We will write $\mathrm{h}^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ for γ in the above proposition. By the same token as in (3.19) we have that for any $d \in D$,

$$((\gamma_4 \underset{2}{\div} \gamma_3) \dot{-} (\gamma_2 \underset{2}{\div} \gamma_1))(d) = \mathrm{h}^2_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(0, 0, 0, 0, 0, 0, d) \tag{3.24}$$

We will write $E[3]$ for $D^7\{(1, 6), (2, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (1, 4), (1, 5), (2, 4), (2, 5)\}$. We define functions ι_1^3, ι_2^3, ι_3^3, and ι_4^3 from D^3 to $E[3]$ as follows:

$$\iota_1^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, 0, 0) \tag{3.25}$$
$$\iota_2^3(d_1, d_2, d_3) = (d_1, d_2, d_3, 0, 0, d_1d_2, 0) \tag{3.26}$$
$$\iota_3^3(d_1, d_2, d_3) = (0, 0, d_3, d_1, d_2, 0, 0) \tag{3.27}$$
$$\iota_4^3(d_1, d_2, d_3) = (0, 0, d_3, d_1, d_2, d_1d_2, d_1d_2d_3) \tag{3.28}$$

By the same token as in Proposition 3.2 we have:

Proposition 3.4. For any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathsf{T}^3(M)$, if the expression $(\gamma_4 \underset{3}{\div} \gamma_3) \dot{-} (\gamma_2 \underset{3}{\div} \gamma_1)$ is well defined, then there exists unique $\gamma \in \mathsf{T}^{E[3]}(M)$ such that $\gamma \circ \iota_i^3 = \gamma_i$ $(i = 1, 2, 3, 4)$.

We will write $\mathrm{h}^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}$ for γ in the above proposition. By the same token as in (3.19) we have that for any $d \in D$,

$$((\gamma_4 \underset{3}{\div} \gamma_3) \dot{-} (\gamma_2 \underset{3}{\div} \gamma_1))(d) = \mathrm{h}^3_{(\gamma_1,\gamma_2,\gamma_3,\gamma_4)}(0, 0, 0, 0, 0, 0, d) \tag{3.29}$$

The crucial step in the proof of Theorem 2.7 is epitomized by the following theorem.

Theorem 3.5. The diagram consisting of objects

(3.30) $E[1]$, $E[2]$, $E[3]$
(3.31) H_{12}, H_{23}, H_{31}, all of which are equal to $D^3 \oplus D^3$
(3.32) G, which is equal to $D^8\{(2, 4), (3, 4), (1, 5), (3, 5), (1\ 6), (2, 6), (4, 5), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (1, 8), (2, 8), (3, 8), (4, 8), (5, 8), (6, 8), (7, 8)\}$

and of morphisms

(3.33) $h_{12}^1: H_{12} \to E[1]$, $h_{12}^2: H_{12} \to E[2]$, $h_{23}^2: H_{23} \to E[2]$, $h_{23}^3: H_{23} \to E[3]$, $h_{31}^3: H_{31} \to E[3]$, $h_{31}^1: H_{31} \to E[1]$
(3.34) $k_1: E[1] \to G$, $k_2: E[2] \to G$, $k_3: E[3] \to G$

is a quasi-colimit diagram of small objects, where

(3.35) $h^1_{12} = \iota^1_2 \oplus \iota^1_3$, $h^2_{12} = \iota^2_4 \oplus \iota^2_1$, $h^2_{23} = \iota^2_2 \oplus \iota^2_3$, $h^3_{23} = \iota^3_4 \oplus \iota^3_1$,
$h^3_{31} = \iota^3_2 \oplus \iota^3_3$, $h^1_{31} = \iota^1_4 \oplus \iota^1_1$

(3.26) $k_1(d_1, d_2, d_3, d_4, d_5, d_6, d_7)$
$= (d_1, d_2 + d_4, d_3 + d_5, d_6 - d_2d_3 - d_4d_5, -d_1d_5, d_1d_4,$
$d_7 + d_1d_2d_3, d_1d_2d_3)$
for any $(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[1]$

(3.27) $k_2(d_1, d_2, d_3, d_4, d_5, d_6, d_7)$
$= (d_1 + d_5, d_2, d_3 + d_4, -d_2d_3, d_6 - d_1d_3 - d_4d_5, d_1d_2,$
$d_2d_4d_5, d_7)$
for any $(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[2]$

(3.38) $k_3(d_1, d_2, d_3, d_4, d_5, d_6, d_7)$
$= (d_1 + d_4, d_2 + d_5, d_3, -d_4d_5, -d_1d_3, d_6, -d_7, -d_7, + d_3d_4d_5)$
for any $(d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in E[3]$

Proof. The so-called general Kock axiom warrants that functions γ_1: $E[1] \to \mathbb{R}$, γ_2: $E[2] \to \mathbb{R}$, γ_3: $E[3] \to \mathbb{R}$, and γ: $G \to \mathbb{R}$ should be polynomials of infinitesimals in D with coefficients in $\mathbb{R}$ of the following forms:

(3.39) $\gamma_1(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^1 + a^1_1d_1 + a^1_2d_2 + a^1_3d_3$
$+ a^1_4d_4 + a^1_5d_5 + a^1_6d_6 + a^1_7d_7 + a^1_{12}d_1d_2 + a^1_{13}d_1d_3$
$+ a^1_{14}d_1d_4 + a^1_{15}d_1d_5 + a^1_{16}d_1d_6 + a^1_{23}d_2d_3 + a^1_{45}d_4d_5$
$+ a^1_{123}d_1d_2d_3 + a^1_{145}d_1d_4d_5$

(3.40) $\gamma_2(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^2 + a^2_1d_1 + a^2_2d_2 + a^2_3d_3$
$+ a^2_4d_4 + a^2_5d_5 + a^2_6d_6 + a^2_7d_7 + a^2_{12}d_1d_2 + a^2_{13}d_1d_3$
$+ a^2_{23}d_2d_3 + a^2_{24}d_2d_4 + a^2_{25}d_2d_5 + a^2_{26}d_2d_6 + a^2_{45}d_4d_5$
$+ a^2_{123}d_1d_2d_3 + a^2_{245}d_2d_4d_5$

(3.41) $\gamma_3(d_1, d_2, d_3, d_4, d_5, d_6, d_7) = a^3 + a^3_1d_1 + a^3_2d_2 + a^3_3d_3$
$+ a^3_4d_4 + a^3_5d_5 + a^3_6d_6 + a^3_7d_7 + a^3_{12}d_1d_2 + a^3_{13}d_1d_3$
$+ a^3_{23}d_2d_3 + a^3_{34}d_3d_4 + a^3_{35}d_3d_5 + a^3_{36}d_3d_6 + a^3_{45}d_4d_5$
$+ a^3_{123}d_1d_2d_3 + a^3_{345}d_3d_4d_5$

(3.42) $\gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) = b + b_1d_1 + b_2d_2$
$+ b_3d_3 + b_4d_4 + b_5d_5 + b_6d_6 + b_7d_7 + b_8d_8 + b_{12}d_1d_2$
$+ b_{13}d_1d_3 + b_{14}d_1d_4 + b_{23}d_2d_3 + b_{25}d_2d_6 + b_{36}d_3d_6$

It is easy to see that for any $(d_1, d_2, d_3, d_4, d_5, d_6) \in D^3 \oplus D^3$,

(3.43) $(\gamma_1 \circ h^1_{12})(d_1, d_2, d_3, d_4, d_5, d_6)$
$= a^1 + a^1_1d_1 + a^1_2d_2 + a^1_3d_3 + a^1_6d_2d_3 + a^1_{12}d_1d_2$
$+ a^1_{13}d_1d_3 + a^1_{16}d_1d_2d_3 + a^1_{23}d_2d_3 + a^1_{123}d_1d_2d_3$
$+ a^1_1d_4 + a^1_4d_5 + a^1_5d_6 + a^1_{14}d_4d_5 + a^1_{15}d_4d_6 + a^1_{45}d_5d_6$
$+ a^1_{145}d_4d_5d_6$
$= a^1 + a^1_1d_1 + a^1_2d_2 + a^1_3d_3 + a^1_{12}d_1d_2 + a^1_{13}d_1d_3$
$+ (a^1_6 + a^1_{23})d_2d_3 + (a^1_{16} + a^1_{123})d_1d_2d_3 + a^1_1d_4$
$+ a^1_4d_5 + a^1_5d_6 + a^1_{14}d_4d_5 + a^1_{15}d_4d_6 + a^1_{45}d_5d_6$
$+ a^1_{145}d_4d_5d_6$

$$
\begin{aligned}
(3.44)\quad & (\gamma_2 \circ h^2_{12})(d_1, d_2, d_3, d_4, d_5, d_6) \\
& = a^2 + a^2_2 d_2 + a^2_4 d_3 + a^2_5 d_1 + a^2_6 d_1 d_3 + a^2_7 d_1 d_2 d_3 \\
& + a^2_{24} d_2 d_3 + a^2_{25} d_1 d_2 + a^2_{26} d_1 d_2 d_3 + a^2_{45} d_1 d_3 \\
& + a^2_{245} d_1 d_2 d_3 + a^2_1 d_4 + a^2_2 d_5 + a^2_3 d_6 + a^2_{12} d_4 d_5 \\
& + a^2_{13} d_4 d_6 + a^2_{23} d_5 d_6 + a^2_{123} d_4 d_5 d_6 \\
& = a^2 + a^2_5 d_1 + a^2_2 d_2 + a^2_4 d_3 + a^2_{25} d_1 d_2 \\
& + (a^2_6 + a^2_{45}) d_1 d_3 + a^2_{24} d_2 d_3 \\
& + a^2_7 + a^2_{26} + a^2_{245}) d_1 d_2 d_3 + a^2_1 d_4 + a^2_2 d_5 + a^2_3 d_6 \\
& + a^2_{12} d_4 d_5 + a^2_{13} d_4 d_6 + a^2_{23} d_5 d_6 + a^2_{123} d_4 d_5 d_6
\end{aligned}
$$

Therefore the condition that $\gamma_1 \circ h^1_{12} = \gamma_2 \circ h^2_{12}$ is equivalent to the following conditions as a whole:

$$
\begin{aligned}
(3.45)\quad & a^1 = a^2 \\
(3.46)\quad & a^1_1 = a^2_5,\ a^1_2 = a^2_2,\ a^1_3 = a^2_4,\ a^1_1 = a^2_1,\ a^1_4 = s^2_2,\ a^1_5 = a^2_3 \\
(3.47)\quad & a^1_{12} = a^2_{25},\ a^1_{13} = a^2_6 + a^2_{45},\ a^1_6 + a^1_{23} = a^2_{24},\ a^1_{14} = a^2_{12}, \\
& a^1_{15} = a^2_{13},\ a^1_{45} = a^2_{23} \\
(3.48)\quad & a^1_{16} + a^1_{123} = a^2_7 + a^2_{26} + a^2_{245},\ a^1_{145} = a^2_{123}
\end{aligned}
$$

By the same token the condition that $\gamma_2 \circ h^2_{23} = \gamma_3 \circ h^2_{23}$ is equivalent to the following conditions as a whole:

$$
\begin{aligned}
(3.49)\quad & a^2 = a^3 \\
(3.50)\quad & a^2_2 = a^3_5,\ a^2_3 = a^3_3,\ a^2_1 = a^3_4,\ a^2_2 = a^3_2,\ a^2_4 = a^3_3,\ a^2_5 = a^3_1 \\
(3.51)\quad & a^2_{23} = a^3_{35},\ a^2_{12} = a^3_6 + a^3_{45},\ a^2_6 + a^2_{13} = a^3_{34},\ a^2_{24} = a^3_{23}, \\
& a^2_{25} = a^3_{12},\ a^2_{45} = a^3_{13} \\
(3.52)\quad & a^2_{26} + a^2_{123} = a^3_7 + a^3_{36} + a^3_{345},\ a^2_{245} = a^3_{123}
\end{aligned}
$$

By the same token again the condition that $\gamma_3 \circ h^3_{31} = \gamma_1 \circ h^1_{31}$ is equivalent to the following conditions as a whole:

$$
\begin{aligned}
(3.53)\quad & a^3 = a^1 \\
(3.54)\quad & a^3_3 = a^1_5,\ a^3_1 = a^1_1,\ a^3_2 = a^1_4,\ a^3_3 = a^1_3,\ a^3_4 = a^1_1,\ a^3_5 = a^1_2 \\
(3.55)\quad & a^3_{13} = a^1_{15},\ a^3_{23} = a^1_6 + a^1_{45},\ a^3_6 + a^3_{12} = a^1_{14},\ a^3_{34} = a^1_{13}, \\
& a^3_{35} = a^1_{23},\ a^3_{45} = a^1_{12} \\
(3.56)\quad & a^3_{36} + a^3_{123} = a^1_7 + a^1_{16} + a^1_{145},\ a^3_{345} = a^1_{123}
\end{aligned}
$$

Three conditions (3.45), (3.49), and (3.53) can be combined into

$$
(3.57)\quad a^1 = a^2 = a^3
$$

Three conditions (3.46), (3.50), and (3.54) are to be superseded by the following three conditions as a whole:

$$
\begin{aligned}
(3.58)\quad & a^1_1 = a^2_1 = a^3_1 = a^2_5 = a^3_4 \\
(3.59)\quad & a^1_2 = a^2_2 = a^3_2 = a^1_4 = a^3_5 \\
(3.60)\quad & a^1_3 = a^2_3 = a^3_3 = a^1_5 = a^2_4
\end{aligned}
$$

Three conditions (3.47), (3.51), and (3.55) are equivalent to the following six conditions as a whole:

(3.61) $a_{12}^1 = a_{12}^2 = a_{12}^3$
(3.62) $a_{13}^1 = a_{13}^2 = a_{13}^3$
(3.63) $a_{23}^1 = a_{23}^2 = a_{23}^3$
(3.64) $a_{14}^1 = a_{12}^1 + a_6^3,\ a_{15}^1 = a_{13}^1 - a_6^2,\ a_{45}^1 = a_{23}^1$
(3.65) $a_{24}^2 = a_{23}^2 + a_6^1,\ a_{25}^2 = a_{12}^2 - a_6^3,\ a_{45}^2 = a_{13}^3$
(3.66) $a_{34}^3 = a_{13}^3 + a_6^2,\ a_{35}^3 = a_{23}^3 - a_6^1,\ a_{45}^3 = a_{12}^3$

Conditions (3.48), (3.52), and (3.56) imply that

(3.67) $$\begin{aligned} & a_7^1 + a_7^2 + a_7^3 \\ &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{145}^1) + (a_{16}^1 + a_{123}^1 - a_{26}^2 \\ &\quad - a_{245}^2) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{345}^3) \\ &= (a_{36}^3 + a_{123}^3 - a_{16}^1 - a_{123}^2) + (a_{16}^1 + a_{123}^1 - a_{26}^2 \\ &\quad - a_{123}^3) + (a_{26}^2 + a_{123}^2 - a_{36}^3 - a_{123}^1) \\ &= 0 \end{aligned}$$

Therefore conditions (3.48), (3.52), and (3.56) are to be replaced by the following conditions as a whole:

(3.68) $a_{145}^1 - a_{123}^1 = a_7^3 + a_{36}^3 - a_{26}^2$
(3.69) $a_{245}^2 - a_{123}^2 = a_7^1 + a_{16}^1 - a_{36}^3$
(3.70) $a_{345}^3 - a_{123}^3 = a_7^2 + a_{26}^2 - a_{16}^1$
(3.71) $a_{145}^1 = a_{123}^2$ and $a_{245}^2 = a_{123}^3$
(3.72) $a_7^1 + a_7^2 + a_7^3 = 0$

Indeed the condition that $a_{345}^3 = a_{123}^1$ is derivative from the above five conditions:

(3.73) $$\begin{aligned} & a_{345}^3 \\ &= a_{123}^3 + a_7^2 + a_{26}^2 - a_{16}^1 \quad [(3.70)] \\ &= a_{245}^2 + a_7^2 + a_{26}^2 - a_{16}^1 \quad [(3.71)] \\ &= a_{123}^2 + a_7^1 - a_{36}^3 + a_7^2 + a_{26}^2 \quad [(3.69)] \\ &= a_{145}^1 + a_7^1 - a_{36}^3 + a_7^2 + a_{26}^2 \quad [(3.71)] \\ &= a_{123}^1 + a_7^1 + a_7^2 + a_7^3 \quad [(3.68)] \\ &= a_{123}^1 \quad [(3.72)] \end{aligned}$$

Now it is not difficult to see that $\gamma_1 \circ h_{12}^1 = \gamma_2 \circ h_{12}^2$, $\gamma_2 \circ h_{23}^2 = \gamma_3 \circ h_{23}^3$, and $\gamma_3 \circ h_{31}^3 = \gamma_1 \circ h_{31}^1$ exactly when there exists $\gamma: G \to \mathbb{R}$ with $\gamma_i = \gamma \circ k_i$ $(i = 1, 2, 3)$, in which γ is of the following form:

(3.74) $$\begin{aligned} & \gamma(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8) = a^1 + a_1^1 d_1 + a_2^1 d_2 \\ &\quad + a_3^1 d_3 + a_6^1 d_4 + a_6^2 d_5 + a_6^3 d_6 + a_7^1 d_7 + a_7^2 d_8 + a_{12}^1 d_1 d_2 \\ &\quad + a_{13}^1 d_1 d_3 + a_{16}^1 d_1 d_4 + (a_{23}^2 + a_6^1) d_2 d_3 + a_{26}^2 d_2 d_5 \\ &\quad + a_{36}^3 d_3 d_6 \end{aligned}$$

This completes the proof of the theorem. ■

As a direct corollary of Theorem 3.5 we have:

Theorem 3.6. For any γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312}, $\gamma_{321} \in \mathsf{T}^3(M)$, if all expressions (2.9)–(2.11) are well defined, then there exists a unique $\gamma \in \mathsf{T}^G(M)$ such that

$$\gamma \circ k_1 = \mathtt{h}^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})} \quad (3.75)$$
$$\gamma \circ k_2 = \mathtt{h}^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})} \quad (3.76)$$
$$\gamma \circ k_3 = \mathtt{h}^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})} \quad (3.77)$$

Proof. Since

$$\begin{aligned}(3.78)\quad & \mathtt{h}^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})} \circ h^1_{12} \\ &= \mathtt{l}_{(\gamma_{231},\gamma_{132})} \\ &= \mathtt{h}^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})} \circ h^2_{12} \\ (3.79)\quad & \mathtt{h}^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})} \circ h^2_{23} \\ &= \mathtt{l}_{(\gamma_{312},\gamma_{213})} \\ &= \mathtt{h}^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})} \circ h^3_{23} \\ (3.80)\quad & \mathtt{h}^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})} \circ h^3_{31} \\ &= \mathtt{l}_{(\gamma_{123},\gamma_{321})} \\ &= \mathtt{h}^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})} \circ h^1_{31}\end{aligned}$$

the desired conclusion follows directly from Theorem 3.5. ■

We will write $\mathtt{m}_{(\gamma_{123},\gamma_{132},\gamma_{213},\gamma_{231},\gamma_{312},\gamma_{321})}$ or $\mathtt{m}$ for short for the above γ.

Once the above theorem is established, we can proceed in the same lines as in Nishimura (1997, pp. 1117–1118) so as to get the general Jacobi identity. Indeed we note that for any $d \in D$,

$$\begin{aligned}(3.81)\quad & ((\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}) \dot{-} (\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}))(d) \\ &= \mathtt{h}^1_{(\gamma_{321},\gamma_{231},\gamma_{132},\gamma_{123})}(0, 0, 0, 0, 0, 0, d) \quad [(3.19)] \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, d, 0) \\ (3.82)\quad & ((\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}) \dot{-} (\gamma_{312} \underset{2}{\dot{-}} \gamma_{132}))(d) \\ &= \mathtt{h}^2_{(\gamma_{132},\gamma_{312},\gamma_{213},\gamma_{231})}(0, 0, 0, 0, 0, 0, d) \quad [(3.24)] \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, 0, d) \\ (3.83)\quad & ((\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}) \dot{-} (\gamma_{123} \underset{3}{\dot{-}} \gamma_{213}))(d) \\ &= \mathtt{h}^3_{(\gamma_{213},\gamma_{123},\gamma_{321},\gamma_{312})}(0, 0, 0, 0, 0, 0, d) \quad [(3.29)] \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, -d, -d)\end{aligned}$$

Therefore, letting t_1, t_2, and t_3 denote expressions (2.9)–(2.11) in order, we have

$$\begin{aligned}(3.84)\quad & \mathtt{l}_{(t_1,t_2,t_3)}(d_1, d_2, d_3) \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, d_1 - d_3, d_2 - d_3) \text{ for any} \\ & (d_1, d_2, d_3) \in D(3)\end{aligned}$$

This means that for any $d \in D$,

$$(3.85) \quad \begin{aligned} &(t_1 + t_2 + t_3)(d) \\ &= \mathtt{l}_{(t_1,t_2,t_3)}(d, d, d) \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, d - d, d - d) \\ &= \mathtt{m}(0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

This completes the proof of Theorem 2.7. ■

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